Final Exam — Group Theory (WIGT-07)

Monday January 21, 2019, 9:00h–12.00h

University of Groningen

Instructions

1. Write your name and student number on every page you hand in.
2. All answers need to be accompanied with an explanation or a calculation.
3. Your grade for this exam is $(P + 10)/10$, where P is the number of points for this exam.

Problem 1 (15 points)

a) Give the definition of a homomorphism of groups.

Solution: Let $(G_1, \cdot, e_1)$ and $(G_2, *, e_2)$ be groups. A *homomorphism* from G_1 to G_2 is a map $f : G_1 \to G_2$ satisfying $f(x \cdot y) = f(x) * f(y)$ for all $x, y \in G_1$. (5 points)

b) Write down the structure theorem for finitely generated abelian groups.

Solution: For any finitely generated abelian group there exist a unique integer $r \geq 0$ and a unique (possibly empty) finite sequence $(d_1, \ldots, d_m)$ of integers $d_i > 1$ satisfying $d_m | d_{m-1} | \ldots | d_1$, such that

$$A \cong \mathbb{Z}^r \times \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_m\mathbb{Z}.$$

(5 points)

c) Give the definition of the conjugacy class of an element of a group.

Solution: Let G be a group. The *conjugacy class* of $x \in G$ is

$$C_x = \left\{ y \in G \mid \text{there exists } a \in G \text{ with } axa^{-1} = y \right\}.$$

(5 points)

Problem 2 (15 points)

Let $\tau = (1\,5\,8\,7\,3)(2\,4\,7\,3\,6)(1\,6\,7\,4\,9\,3) \in S_9$.

a) Compute the order of τ.

Solution: We compute the decomposition of τ into disjoint cycles and find $\tau = (1\,2\,4\,9\,6) \circ (3\,5\,8\,7)$ (3 points). Hence $\mathrm{ord}(\tau) = \mathrm{lcm}(5, 4) = 20$ (1 point), because the order of a product of disjoint cycles is the least common multiple of the lengths of the cycles. (1 point) (5 points in total)

b) Compute the sign of τ.

Solution: By multiplicativity of the sign or by using the formula for the sign of a product of cycles (1 point), we find it is $(-1)^{5-1+5-1+6-1} = -1$ (using the definition of τ) or $(-1)^{5-1+4-1} = -1$ (using the decomposition into disjoint cycles). (2 points) (3 points in total)

c) Find the number of elements of the conjugacy class of τ.

Solution: The conjugacy class of a permutation is determined completely by the decomposition into disjoint cycles, so the desired number of elements is the number of $\sigma = (i_1\, i_2\, i_3\, i_4\, i_5) \circ (j_1\, j_2\, j_3\, j_4) \in S_9$, where the two cycles are disjoint. (2 points) The number of k-cycles in S_n is $\frac{n!}{k(n-k)!}$ (1 point). Hence there are $\frac{9!}{5(9-5)!} = 9 \cdot 8 \cdot 7 \cdot 6$ choices for $(i_1\, i_2\, i_3\, i_4\, i_5)$ (2 points). After these are fixed, there are $4!/4 = 6$ choices for $(k_1\, k_2\, k_3\, k_4)$. (1 point) Therefore the conjugacy class of τ contains $9 \cdot 8 \cdot 7 \cdot 6^2 = 18144$ elements (1 point). (7 points in total)

Problem 3 (18 points)

Let G and G' be groups and let $f\colon G \to G'$ be a homomorphism.

(a) Show that if $H' \leq G'$ is a normal subgroup, then the preimage

$$f^{-1}(H') = \{x \in G \,:\, f(x) \in H'\}$$

is a normal subgroup of G (you do not have to show that $f^{-1}(H')$ is a subgroup of G).

Solution: Let $H := f^{-1}(H')$. Let $h \in H$, then $f(h) \in H'$ by definition of H (1 point). For $a \in G$ we have $f(aha^{-1}) = f(a)f(h)f(a^{-1})$ (1 point) since f is a homomorphism (0.5 points). But H' is normal, therefore $f(a)f(h)f(a^{-1}) \in H'$ (1 point), which implies that $aha^{-1} \in H$ (0.5 points). This proves that H is normal. (1 point) (5 points in total)

(b) Show that if f is surjective and if $H \leq G$ is a normal subgroup, then the image $f(H)$ is a normal subgroup of G' (you do not have to show that $f(H)$ is a subgroup of G').

Solution: Let $H' := f(H)$. Let $h' \in H'$, then there is some $h \in H$ such that $h' = f(h)$ by definition of H' (1 point). As f is surjective, every $a' \in G'$ has a preimage $a \in G$ under f (1 point). Therefore $a'h'a'^{-1} = f(a)f(h)f(a)^{-1} = f(a)f(h)f(a^{-1}) = f(aha^{-1})$ (1.5 points) because f is a homomorphism (0.5 points). But H is normal, so $aha^{-1} \in H$ (1 point), and we conclude $f(aha^{-1}) \in H'$ and $a'h'a'^{-1} \in H'$ (1 point), which implies that H' is normal. (1 point) (7 points in total)

(c) Find an example of groups G and G' such that there is a homomorphism $f : G \to G'$ and a normal subgroup H of G with the property that $f(H)$ is not a normal subgroup of G'.

Solution: Let $G' = S_3$ and $H' = \langle (1\,2) \rangle = \{(1), (1\,2)\} \leq G'$. (1 point) Then H' is not a normal subgroup of G'. To see this, take for instance $\tau = (2\,3) = \tau^{-1}$, then $\tau \circ (1\,2) \circ \tau^{-1} = (1\,3) \notin H'$. (2 points) On the other hand, H' is a group in its own right. So we take $G := H'$ and we let H be the subgroup $H := G$ of G. (1 point) Then H is trivially normal in G, (1 point) but $f(H) = H'$ is not normal in G' if we take for f the inclusion homomorphism, i.e. $f((1)) = (1)$ and $f((1\,2)) = (1\,2)$. (1 point) This example can be generalized in an obvious way. (6 points in total)

Problem 4 (10 points)

Let G be a group of order 56. Show that G is not simple.

Solution: We have $56 = 2^3 \cdot 7$. (1 point) For a prime $p \mid 56$, let $n_p = n_p(G)$ be the number of Sylow-p groups in G. If we find $n_p = 1$ for some p, then we know that the unique Sylow p-group in G is normal and, since it has order p, it is not G or $\{e\}$, so G is not simple. (2 points)

By Sylow's theorem $n_7 \equiv 1 \pmod 7$ and $n_7 \mid 8$, so $n_7 \in \{1, 8\}$ (1 point). Suppose $n_7 = 8$; it suffices to show that $n_2 = 1$ (1 point). If H, H' are distinct 7-Sylow groups in G, then their intersection consists only of the unit element e (for instance, since their intersection is a subgroup of H, so by Lagrange it has order dividing 7). Hence there are $8 \cdot (7 - 1) = 48$ elements of order 7 in G, as every element $\neq e$ of a 7-Sylow group has order 7. (3 points) This implies $n_2 = 1$, because $n_2 \geq 1$ and every Sylow-2 group consists of precisely 8 elements, none of which are of order 7. (2 points)

Problem 5 (12 points)

Let G be the abelian group $(\mathbb{R}^3, +, 0)$ and let $L = \{(x_1, x_2, x_3) \in G : x_1 + x_2 = 0 = x_3\}$.

(a) Show that L is a normal subgroup of G.

Solution:

In order to use the subgroup criterion (1 point), we check that

(H1) the unit element 0 is in G, which is obvious (1 point)

(H2) if $x, y \in G$, we have $z := x + y \in G$, since

$$z_1 + z_2 = x_1 + y_1 + x_2 + y_2 = x_1 + x_2 + y_1 + y_2 = 0 = x_3 + y_3 = z_3; \text{ (1 point)}$$

(H3) if $x \in G$ we have $w := -x \in G$, since $w_1 + w_2 = -(x_1 + x_2) = 0 = -x_3 = w_3$ (1 point).

Any subgroup of the abelian group G must be normal itself (1 point).

A simpler solution is to use that L is the kernel of the homomorphism ϕ in (b), since the kernel of a homomorphism is always a normal subgroup (5 points).

(b) Show that $G/L \cong \mathbb{R}^2$.

Solution: Consider the map

$$\phi\colon G \to \mathbb{R}^2 \; ; \quad x := (x_1, x_2, x_3) \mapsto (x_1 + x_2, x_3).$$

Then ϕ is a homomorphism, since for $x, y \in G$ we have

$$\begin{aligned} \phi(x + y) &= (x_1 + y_1 + x_2 + y_2, x_3 + y_3) \\ &= (x_1 + x_2, x_3) + (y_1 + y_2, y_3) \\ &= \phi(x) + \phi(y) \end{aligned}$$

(2 points) It is clear that $\ker \phi = L$ (1 point). The homomorphism ϕ is surjective, because if $z = (z_1, z_2) \in \mathbb{R}^2$, then we have $\phi(x) = (z_1 + 0, z_2) = z$ for $x = (z_1, 0, z_2) \in G$. (2 points) Therefore the homomorphism theorem implies

$$G/L = G/\ker \phi \cong \phi(G) = \mathbb{R}^2.$$

(2 points)

Problem 6 (20 points)

Determine the rank and the elementary divisors of each of the following groups:

a) $\mathbb{Z}^3 \times 15\mathbb{Z} \times (\mathbb{Z}/5\mathbb{Z})^\times \times \mathbb{Z}/14\mathbb{Z}$.

Solution: We have $\mathbb{Z} \cong 15\mathbb{Z}$ (1 point) via the isomorphism $x \mapsto 15x$ (0.5 points). Since $(\mathbb{Z}/5\mathbb{Z})^\times \times \mathbb{Z}/14\mathbb{Z}$ is finite, this shows that the rank is 4 (1.5 points).

The group $(\mathbb{Z}/5\mathbb{Z})^\times$ has order 4, so every element has order $1, 2$ or 4. Since $\bar{2}^2 = \bar{4} \neq \bar{1}$, the order of $\bar{2}$ must be 4, which implies $(\mathbb{Z}/5\mathbb{Z})^\times \cong \mathbb{Z}/4\mathbb{Z}$ (2 points). By the Chinese remainder theorem,

$$\begin{aligned}(\mathbb{Z}/5\mathbb{Z})^\times \times \mathbb{Z}/14\mathbb{Z} &\cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z} \\ &\cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\ &\cong \mathbb{Z}/28\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \quad \text{(2 points)}\end{aligned}$$

so the elementary divisors are 2 and 28 (1 point). (8 points in total)

b) $\mathbb{Z}^3/H$, where $H \leq \mathbb{Z}^3$ is generated by $(3, 1, 2), (-4, 6, 2), (-1, 7, 4)$.

Solution: Let A denote the matrix with rows equal to the given generators of H. We apply the algorithm from the lecture to transform A into a diagonal matrix whose diagonal entries are the elementary divisors (and possibly 0's and 1's) (2 points):

$$\begin{pmatrix} 3 & 1 & 2 \\ -4 & 6 & 2 \\ -1 & 7 & 4 \end{pmatrix} \sim \begin{pmatrix} -1 & 7 & 4 \\ -4 & 6 & 2 \\ 3 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} -1 & 7 & 4 \\ 0 & -22 & -14 \\ 0 & 22 & 14 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 0 \\ 0 & -22 & -14 \\ 0 & 22 & 14 \end{pmatrix} \text{ (2 points)}$$

We continue with the bottom right 2×2 matrix (1 point):

$$\begin{aligned}\begin{pmatrix} -22 & -14 \\ 22 & 14 \end{pmatrix} &\sim \begin{pmatrix} -14 & -22 \\ 14 & 22 \end{pmatrix} \sim \begin{pmatrix} -14 & -22 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -14 & 6 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 6 & -14 \\ 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 6 & -2 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -2 & 6 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \text{ (2 points)}\end{aligned}$$

Hence the desired matrix is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ (1 point)}$$

It follows that $\mathbb{Z}^3/H \cong \mathbb{Z} \times \mathbb{Z}/1\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (2 points), so the rank is 1 (1 point) and the only elementary divisor is 2 (1 point) (12 points in total).

End of test (90 points)